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A remark on convergence of orthogonal series

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1. Let S denote the set of Lebesgue measurable, almost everywhere finite functions on the interval (0, 1). Let $T=\|t_{i,j}\|_0^\infty$ be a matrix such that

$$(1)\qquad |t_{i,j}|\leqq K(<\infty)\quad (i,j=0,1,\ldots),\quad \lim_{i\to\infty} t_{i,j}=1\quad (j=0,1,\ldots),$$

and let $f=\{f_k(x)\}_0^\infty$ be a sequence of functions belonging to S. A series

$$(2)\qquad \sum_{k=0}^{\infty} c_k f_k(x)$$

is said to be T summable in measure (almost everywhere) if the series

$$t_i(x)=\sum_{k=0}^{\infty} t_{i,k} c_k f_k(x)\quad (i=0,1,\ldots)$$

converge in measure (almost everywhere) and the sequence $\{t_i(x)\}_0^\infty$ converges in measure (almost everywhere) to a function belonging to S.

The system f is said to be a T convergence system in measure (T convergence system) for l_2 if for every $c=\{c_k\}_0^\infty\in l_2$ the series (2) is T summable in measure (T summable almost everywhere).

The system f is said to be a convergence system in measure (almost everywhere) for l_2 if $c\in l^2$ implies the convergence of the series (2) in measure (almost everywhere).

Joó [3] proved a general theorem which contains the following statement as a special case:

Let T be a matrix satisfying conditions (1). *If the system f is a T convergence system in measure for l_2, then it is also a convergence system in measure for l_2.*

2. A natural question is whether a similar statement is true for almost everywhere convergence.

In this note we give a negative answer to this question.

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Let $\nu=\{\nu_n\}_0^\infty$ be a strictly increasing sequence of non-negative integers, $\nu_0=0$. We call T_ν the summation process generated by a matrix $\|t_{i,k}\|$ of the form

$$t_{ik}=1 \quad (k=0,1,\dots,\nu_i), \quad t_{i,k}=0 \quad (k=\nu_i+1,\nu_i+2,\dots) \quad (i=0,1,\dots).$$

The T summation is said to be equivalent to T_ν summation if for every $c\in l_2$ and for every orthonormal system $\varphi=\{\varphi_k(x)\}_0^\infty$ on $(0,1)$ the orthogonal series

$$\sum_{k=0}^{\infty} c_k\varphi_k(x) \tag{3}$$

is T summable almost everywhere if and only if it is T_ν summable almost everywhere. (We recall the fact that, e.g., $(C,1)$ summability is equivalent to $T_{\{2^\nu\}}$ summability; see e.g. ALEXITS [1], p. 118.)

After this preparation, our statement is:

Theorem. *Let ν be a sequence of indices such that $\overline{\lim_{n\to\infty}}\,(\nu_{n+1}-\nu_n)=\infty$. Let T be a summation process equivalent to T_ν. Then there exists an orthonormal system $\Phi=\{\varphi_k(x)\}_0^1$ on $(0,1)$, which is a T convergence system for l_2 but is not a convergence system for l_2, indeed there exists a sequence $c\in l_2$ such that the series* (3) *diverges almost everywhere.*

We remark that the system Φ in our Theorem is obtained by a rearrangement of the Walsh system $\{w_n(x)\}_0^\infty$. Using ideas of F. MÓRICZ [4] it is easy to see that one can obtain an orthonormal system, with similar properties, also by rearrangement of the trigonometrical system $\{1, \cos 2\pi x, \sin 2\pi x, \dots\}$.

3. The proof of the Theorem. Let $r_n(x)=\operatorname{sign}\sin 2^n\pi x$ be the n^{th} Rademacher function $(n=0,1,\dots)$. The Walsh functions are defined as follows. Let $w_0(x)=r_0(x)$. If n is a natural number and $n=2^{k_1}+\dots+2^{k_m}$ $(0\leqq k_1<\dots<k_m;\ k_i$ integers) is its diadic expansion then define

$$w_n(x)=r_{k_1+1}(x)\dots r_{k_m+1}(x).$$

We shall use a Theorem of BILLARD [2] which states that the Walsh system is a convergence system for l_2. We also need the following lemma which is proved essentially in TANDORI [5].

Lemma. *Let $m\geqq 2$ be an arbitrary natural number. Then there exists a sum of the form*

$$S_m(x)=\sum_{k=1}^{l(m)} a_k(m)w_k(x) \quad (l(m)<\mu(m+1)),$$

where $\mu(m)=2^{2^m}$, such that

$$\int_0^1 S_m^2(x)\,dx\leqq\frac{5}{m}, \tag{4}$$

furthermore, it has a rearrangement

$$S_m^*(x) = \sum_{l=1}^{l(m)} a_{k_l(m)}(m)\, w_{k_l(m)}(x)$$

such that

$$\max_{1 \leqq j \leqq l(m)} \left| \sum_{l=0}^{j} a_{k_l(m)}(m)\, w_{k_l(m)}(x) \right| \geqq 1 \quad (x \in (0, 1/4) \setminus D), \tag{5}$$

where D denotes the set of diadic numbers.

Consider the sum

$$\sigma_m(x) = r_{2^{m+1}+1}(x) \cdot S_m(x).$$

According to the definition of Walsh functions, $\sigma_m(x)$ has the form

$$\sigma_m(x) = \sum_{k=\mu(m+1)+1}^{\mu(m+1)+l(m)} b_k(m)\, w_k(x) \quad (l(m) < \mu(m+1)).$$

Our Lemma shows that

$$\int_0^1 \sigma_m^2(x)\, dx \leqq \frac{5}{m}, \tag{6}$$

furthermore, $\sigma_m(x)$ has a rearrangement

$$\sigma_m^*(x) = \sum_{l=1}^{l(m)} b_{k_l(m)}(m)\, w_{k_l(m)}(x),$$

such that

$$\max_{1 \leqq j \leqq l(m)} \left| \sum_{l=1}^{j} b_{k_l(m)}(m)\, w_{k_l(m)}(x) \right| \geqq 1 \quad (x \in (0, 1/4) \setminus D). \tag{7}$$

Now we define the system Φ in our Theorem. First let $\{n_m\}_2^\infty$ be a strictly increasing sequence of indices such that

$$\nu_{n_m+1} - \nu_{n_m} \geqq \mu(m^2+1) \quad (m = 2, 3, \ldots);$$

such a sequence exists according to our assumption concerning ν. For all $m (\geqq 2)$ consider the sum $\sigma_{m^2}(x)$. It is obvious by the definition that in the case $m \neq \overline{m}$ the same Walsh functions do not occur in both $\sigma_{m^2}(x)$ and $\sigma_{\overline{m}^2}(x)$ with coefficients different from zero. Further it is easy to see that the sum $\sigma_{2^2}(x), \ldots, \sigma_{m^2}(x)$ are built from Walsh functions $w_1(x), \ldots, w_{2\mu(m^2+1)-1}(x)$.

Consider the rearrangement of the sum $\sigma_{m^2}(x)$:

$$\sigma_{m^2}^*(x) = \sum_{l=1}^{l(m^2)} b_{k_l(m^2)}(m^2)\, w_{k_l(m^2)}(x).$$

Let

$$\varphi_{\nu_{n_m}+l}(x) = w_{k_l(m^2)}(x) \quad (l = 1, \ldots, l(m^2)).$$

Let

$$\Omega_1 = \bigcup_{m=2}^{\infty} \{k_l(m^2) : l = 1, \ldots, l(m^2)\}, \quad \Omega_2 = \{0, 1, \ldots\} \setminus \Omega_1,$$

and denote the elements of Ω_2 in the order of magnitude by $q_1, q_2, \ldots$. At last let $r_1, r_2, \ldots$ be those indices, in order of magnitude, for which the function $\varphi_k(x)$ are not yet defined. Next let

$$\varphi_{r_i}(x) = w_{q_i}(x) \quad (i = 1, 2, \ldots).$$

So we have defined an orthonormal system $\Phi = \{\varphi_k(x)\}_0^\infty$ on (0, 1), which is a rearrangement of the Walsh system $\{w_k(x)\}_0^\infty$.

Let $c = \{c_k\}_0^\infty \in l_2$ be arbitrary. According to the definition of the functions $\varphi_k(x)$,

$$(8)\qquad \begin{aligned} \sum_{k=0}^{\infty} c_k \varphi_k(x) &= \sum_{i=1}^{\infty} c_{r_i} \varphi_{r_i}(x) + \sum_{m=2}^{\infty} \sum_{j=\nu_{n_m}+1}^{\nu_{n_m}+l(m^2)} c_j \varphi_j(x) = \\ &= \sum_{i=1}^{\infty} c_{r_i} w_{q_i}(x) + \sum_{m=2}^{\infty} \sum_{l=1}^{l(m^2)} c_{\nu_{n_m}+l} w_{k_l(m^2)}(x) = \sum\nolimits_1 + \sum\nolimits_2. \end{aligned}$$

The sum $\sum_1$ is a Walsh expansion in l_2 thus, according to Billard's theorem, it converges almost everywhere on (0, 1).

On the other hand, for all m

$$\sum_{l=1}^{l(m^2)} c_{\nu_{n_m}+l} w_{k_l(m^2)}(x) = \sum_{l=\mu(m^2+1)+1}^{\mu((m+1)^2+1)} \bar{c}_l w_l(x),$$

where

$$\sum_{l=1}^{l(m^2)} c^2_{\nu_{n_m}+l} = \sum_{l=\mu(m^2+1)+1}^{\mu((m+1)^2+1)} \bar{c}_l^2.$$

Now set

$$d_k = c_{\nu^2_{n_m}+j} \quad \text{for} \quad k = \nu_{n_m}+j; \quad j = 1, \ldots, l(m^2), \quad \text{and} \quad d_k = 0 \quad \text{otherwise.}$$

Obviously, $d = \{d_k\}_0^\infty \in l_2$ and the ν_n^{th} partial sum of the series

$$\sum\nolimits_2 = \sum_{k=0}^{\infty} d_k \varphi_k(x)$$

is equal to the $\mu(m^2+1)+l(m^2)^{\text{th}}$ partial sum of the series

$$\sum_{k=0}^{\infty} \bar{c}_k w_k(x)$$

for some m. Apply Billard's theorem again to obtain that the sequence of the ν_n^{th} partial sums of the series $\sum_2$ converges almost everywhere. Using (8) we obtain thath the sequence of the ν_n^{th} partial sums of the series (3) also converges almost everywhere. This shows that the system Φ is a T convergence system for l_2. (We use our assumption for T that it is equivalent to T_ν.)

On the other hand, consider the series

$$(9)\qquad \sum_{m=2}^{\infty} \sigma^2_{m^2}\left(x - \frac{m}{4}\right).$$

From the definition of the system Φ and from (6) it follows that (9) is an l_2-expansion in Φ:

$$\sum_{m=2}^{\infty} \sigma_{m^2}^{*}\left(x-\frac{m}{4}\right) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \quad (\{a_k\}_0^\infty \in l_2).$$

But it is clear from (7) that this series diverges almost everywhere on (0, 1). So our Theorem is proved.

References

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